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Hyperbolicity and Astigmatism

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We study the mechanism of hyperbolicity in high-dimensional Hamiltonian systems. Especially we consider ergodic billiards with focusing components in dimensions $d \geqslant 3$. In this case astigmatism serves as an obstacle to hyperbolicity in billiards with large focusing components. The notion of absolutely focusing mirrors is extended to the dimensions $d \geqslant 3$ and the first classes of ergodic billiards with both focusing and dispersing components are constructed in $d \geqslant 3$.

KEY WORDS: Defocusing; dispersing; astigmatism; absolutely focusing mirrors; ergodicity.

1. INTRODUCTION

Systems with elastic collisions (or billiards) form the most popular and well investigated class of nonuniformly hyperbolic Hamiltonian systems. Hyperbolicity means that locally orbits in a phase space of a dynamical system diverge exponentially. If this phase space is compact then *local* hyperbolicity often generates a *global* chaotic behavior of a system, which is characterized by its ergodicity, mixing (decay of time correlations), etc.

Naturally, smooth, uniformly hyperbolic systems were investigated at first. In this case the rates of instability (Lyapunov exponents) are defined in all points of the phase space and locally do not differ "too much" from each other. Uniformly hyperbolic systems include geodesic flows on manifolds of negative curvature, considered long ago by Hadamard, Hedlund and Hopf. Anosov systems and Smale's axion A systems (see e.g., ref. 1 and references therein) also belong to uniformly hyperbolic dynamical systems.

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The ground-breaking work by Sinai[2] on what is known as Sinai billiards has opened the new era in the studies of hyperbolic systems. Many new tools were introduced in ref. 2 to tackle nonuniformly hyperbolic systems. Sinai billiards appear in many important physical models like e.g., the Lorentz gas. One of the milestones in the studies of hyperbolic billiards was the discovery of the new mechanism of hyperbolicity[3] which differs from the mechanism of hyperbolicity in geodesic flows on manifolds of negative curvature and in Sinai billiards. In the latter systems hyperbolicity is generated by the dispersing character of surfaces of negative curvature or of the dispersing boundary in Sinai billiards. Therefore, this mechanism of hyperbolicity is called the *mechanism of dispersing*. However, the billiards constructed in ref. 3 do not have dispersing components of the boundary. On the contrary, the crucial role is played there by the focusing components, which could be arranged in such way that effectively the trajectories (rays) diverge in phase space because of a strong defocusing. Therefore, this mechanism of hyperbolicity is called the *mechanism of defocusing*. The discovery of this mechanism allowed to construct chaotic geodesic flows on surfaces with nonnegative curvature (e.g., on two-dimensional spheres and tori). A lot of work has been devoted to the study of billiards where hyperbolicity has been generated by the mechanism of defocusing or by both the mechanisms of dispersing and defocusing.[1] However, all these billiards were two-dimensional ones. It should be compared with Sinai billiards which were proved soon after the paper[2] to be chaotic in all dimensions.[1]

The crucial question whether or not the mechanism of defocusing can generate hyperbolicity in dimensions greater than two remained open for a long time until it has been recently settled in refs. 4 and 5. At first, it has been claimed in ref. 6 that the mechanism of defocusing can generate chaotic behavior in $d \geqslant 3$, where some examples of corresponding billiards were also suggested. On the other hand, Wojtkowski[7] constructed some examples of $3d$ billiards with focusing and flat components having linearly stable periodic orbits.

Billiards where hyperbolicity is generated by the mechanism of defocusing occupy the place between dispersing billiards and integrable one (see ref. 1 and Section 4 below). Most importantly the new phenomenon becomes crucial in systems with local focusing in dimensions $d \geqslant 3$. This phenomenon, which is called astigmatism, is well known in the geometric optics.[8] Astigmatism means that a strength of focusing can be quite different in different two-dimensional planes. Especially it can be very weak in some planes while the mechanism of defocusing requires rather strong focusing in all two-dimensional sections. Again, it should be compared with the mechanism of dispersing which just require arbitrary small (but nonzero) dispersing in any two-dimensional plane.

The classes of billiards constructed in refs. 4 and 5, where hyperbolicity is generated by the mechanism of defocusing, allowed only focusing and flat components of the boundary. The main purpose of this paper is to describe some classes of chaotic high-dimensional billiards with all three types (focusing, dispersing and flat) components of the boundary. The other purpose is to formulate the general approach to the construction of hyperbolic billiards in dimensions $d \geqslant 3$ and to extend the notion of absolutely focusing mirrors[9] to the higher dimensions. In the last section we discuss some open problems and a new physical model inspired by the proof that defocusing works in $d \geqslant 3$.

2. LOCAL BEHAVIOR OF BILLIARDS, ASTIGMATISM AND THE MAIN RESULTS

We consider billiards in d-dimensional regions $Q \subset \mathbb{R}^d$ $(d \geqslant 2)$ with piecewise smooth (of class C^3) boundary ∂Q. The boundary ∂Q is equipped with a field of inward unit normal vectors $n(q)$, $q \in \partial Q$. Upon fixation of a unit normal vector at each nonsingular point $q \in \partial Q$ we can define a curvature (the second fundamental form or the curvature operator $K(q)$) at q. We assume that at each regular component $(\partial Q)_i$, $i = 1, 2,..., k$, of the boundary ∂Q the curvature operator is either nonnegative $(K(q) \geqslant 0$, $q \in (\partial Q)_i)$, nonpositive $(K(q) \leqslant 0, q \in (\partial Q)_i)$ or $K(q) = 0$ for any $q \in (\partial Q)_i$. (For $d = 2$ it means that for each component the curvature is positive, negative or identically equal to zero.) By natural reasons we will call such regular components of the boundary ∂Q as dispersing, focusing and flat components, respectively. For instance, a billiard is called dispersing if $K(q) > 0$ at all regular points of the boundary. Our goal is to study billiards which contain at least one strictly focusing component $(\partial Q)_i \in \partial Q$, i.e., $K(q) < 0$ at all regular points of $(\partial Q)_i$.

A billiard is a dynamical system generated by the motion of a point particle with the unit velocity inside the region Q being reflected from its boundary according to the law "the angle of incidence equals the angle of reflection." It means that upon reflection the tangent component of the velocity remains the same, while the normal component changes its sign according to the rule $v_+ = v_- - 2(n(q), v_-)\, n(q)$, where $v_+(v_-)$ is the velocity of the particle immediately after (before) reflection.

The phase space of $\mathcal{M}$ of a billiard is the restriction of the unit tangent bundle of $\mathbb{R}^n$ to Q. We'll use the standard notation for phase points $x = (q, v) \in \mathcal{M}$, where q is the point of the configuration space Q and v is the unit velocity vector. The billiard preserves the Liouville measure $dv = dq\, dw$ where dq and dw are Lebesgue measures on Q and the unit $(d-1)$-dimensional sphere. The corresponding flow will be denoted by

$\{S^t\}$. It is customary for billiard-type systems to study instead of S^t a dynamical system with discrete time, which is called a billiard map T. Denote $M = \{x = (q, v), q \in \partial Q, (v, n(q)) > 0\}$. Let π be the projection of $\mathscr{M}$ onto the configuration space, i.e., $\pi(x) = q$. For $x = (q, v) \in M$ let $\tau(x)$ be the first positive moment of reflection from the boundary of the billiard orbit determined by x. Then $Tx = (q', v') = S^\tau x$, so that q' is the point of the next reflection and v' is the outcoming velocity vector at that point. The billiard map Tpreserves the projection of the Liouville measure to the boundary $d\mu(q, v) = \text{const}(v, n(q))\, dq\, dw$, where dq is the $(d-1)$-dimensional Lebesgue measure on the boundary ∂Q generated by the volume and dw is the $(d-1)$-dimensional Lebesgue (uniform) measure on the unit sphere. The const is the normalizing constant so that $\mu(M) = 1$.

In order to study the local behavior of billiard orbits we introduce a notion of a $(d-1)$-dimensional wave front γ (also called a control surface), which is an infinitesimal surface of class C^2 perpendicular to the orbit. The rate at which the neighboring trajectories diverge is defined by the curvature operator of the wave front γ. Suppose for simplicity that initially the wave front γ is flat, i.e., its curvature operator is identically zero. Then after the reflection from a dispersing component of ∂Q all principle curvatures of γ become positive, which generates the divergence of neighboring trajectories along these hyperplanes (Fig. 1a). However, after the reflection from the focusing components, some of the principle curvatures become negative and thus the neighboring trajectories converge (instead of diverging) along those hyperplanes (Fig. 1b). The mechanism of defocusing ensures that a free path (a path between two consecutive reflections) is long enough so that the corresponding families of trajectories defocus and after that

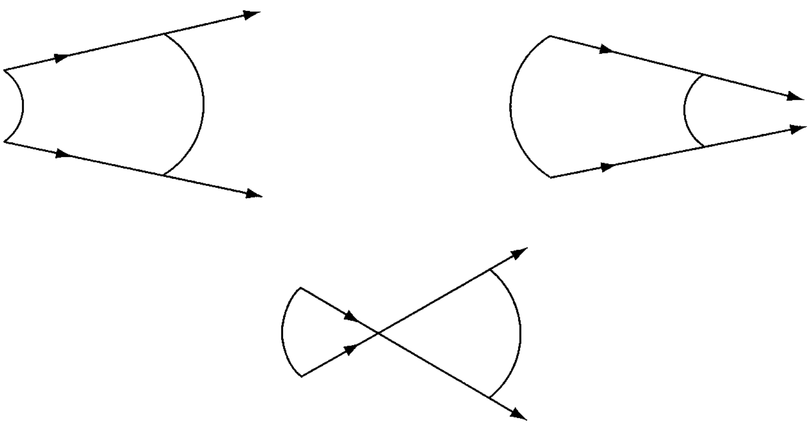

Fig. 1. Dispersing (a), focusing (b) and defocusing (c) beams of rays (trajectories).

diverge, i.e., from a certain point on the free path (the conjugate point) the curvature operator becomes positively definite (Fig. 1c).

Therefore, to understand the dynamics of billiards in the vicinity of the given orbit it is important to know how the curvatures of the control surface γ evolve. Before deriving the corresponding formula we will formulate the first condition on billiards under study.

Condition A. Each focusing component $(\partial Q)_i$ the boundary ∂Q is a spherical cap. By a spherical cap we mean a piece of the $(d-1)$-dimensional sphere bounded by some hyperplane B_i.

Condition A implies that a part of the orbit consisting of consecutive reflections from the same focusing component of the boundary lies in some two-dimensional plane P, which contains the center of the corresponding sphere. The plane P defines the unique direction on a hyperplane U, perpendicular to the orbit. We will call this direction a planar subspace while its orthogonal complement will be called an orthogonal or transversal subspace. Then $U = U_p \oplus U_t$, where $U_p = U \cap P$ and U_t is the $(d-2)$-dimensional orthogonal complement to U_p in U. Since the control surface γ is tangent to U the corresponding planar and the orthogonal directions can be also naturally defined on γ. (Observe, that these directions are not intrinsic notions of the control surface but can be defined only with respect to a given (spherical) focusing component of the boundary ∂Q.)

Now we are ready to describe how the curvatures of the wave front γ evolve in the course of the billiard dynamics (for the detailed description though see refs. 4 and 5). For the sake of simplicity we assume that the principal curvature directions of γ coincide with the planar and orthogonal ones. It is easy to see, that in the planar direction the curvature changes just like in the two-dimensional billiards

$$\kappa_+ = \kappa_- + \frac{2k}{\cos\varphi} \tag{1}$$

where κ_- (κ_+) is the curvature at the moment just before (after) the reflection, φ is the angle of reflection and k is the curvature of the boundary.[2, 3] However, in the orthogonal subspace the curvatures obey [ref. 8, p. 66] another rule

$$\kappa_+^\perp = \kappa_-^\perp + 2k\cos\varphi$$

where $\kappa_-^\perp$ ($\kappa_+^\perp$) is the curvature in the given direction immediately before (after) the reflection and k and φ are the same as in (1).

The relations (1) and (2) show the nature of the astigmatism.[8] It is easy to see that in the planar direction the curvature of the wave front γ always changes upon reflection at least by the value $2k$ while in the orthogonal directions it can change, in principle, by an arbitrarily small value.

Finally, during the free path, the principal curvature directions are preserved. Therefore the curvature evolution in all directions reads as[1, 2]

$$\kappa(t) = \frac{\kappa(0)}{1 + \kappa(0)\, t} \tag{3}$$

It is easy to see from (3) that a focusing (in a given direction) family of trajectories focuses at the moment $t = -1/\kappa(0)$.

The main difficulty in the study of the evolution of the wave front γ is caused by the fact that when γ approaches some spherical cap $C \subset \partial Q$, its principal curvature directions do not generally coincide with the planar and orthogonal directions. To perform the corresponding analysis one must analyze the evolution of the curvature operator of γ (see ref. 4 for details).

The relation (2) shows that, even with very big free path available, along some directions on a control surface trajectory may converge between two consecutive reflections from the boundary (i.e., the case depicted in Fig. 1b arises) and moreover a contraction along some directions may be arbitrarily big. This situation arises because of astigmatism and it does not occur in dimension two. Therefore, chaotic billiards in $2d$ can contain very big focusing component of the boundary, i.e., a focusing component can be almost entire circle.[3] The astigmatism requires to pay some price to ensure hyperbolicity. The next condition represents this price. It says that focusing components of the boundary are not too big.

Let C be a spherical cap of some $(d-1)$-dimensional sphere S^{d-1}, $d \geqslant 3$, and O is the center of this sphere. Take two arbitrary points $Y_1, Y_2 \in \partial C$, where ∂C is the boundary of C. We shall call the angle $Y_1 O Y_2$ a central angle of C corresponding to the points Y_1, Y_2.

Condition B. Let $(\partial Q)_i$ be a focusing component of the boundary ∂Q. Then all central angles of $(\partial Q)_i$ do not exceed 90°.

We also introduce

Condition B′. Let $(\partial Q)_i$ be a focusing component of the boundary ∂Q. Then all central angles of $(\partial Q)_i$ do not exceed 60°.

Now we must take care of free paths to make them sufficiently long in order to allow trajectories defocus after a reflection from a focusing component of the boundary. This will be taken care of by the next three conditions.

Definition 1. A focusing component $(\partial Q)_i$ of the boundary is attached to a single regular component $(\partial Q)_j \subset \partial Q$ if $\partial((\partial Q)_i) \subset (\partial Q)_j$.

Condition C. Each focusing component of the boundary ∂Q is attached to some flat or dispersing component of ∂Q.

Thus a focusing component cannot be attached to another focusing component or to more than one component of ∂Q. The case, when focusing components were attached to flat components of the boundary, was considered in refs. 4 and 5. It is easy to see that the Condition C does not contradict the Conditions A and B. For example there can be spherical dispersing components of ∂Q to which focusing components (spherical caps) are attached.

Let a spherical cap $(\partial Q)_i$ be attached to a regular component $(\partial Q)_j$. Any spherical cap $(\partial Q)_i$ is defined by a $(d-1)$-dimensional sphere S_i^{d-1} and by a hyperplane B_i. We denote by B_i' the hyperplane which is parallel to B_i and contains the center of the sphere S_i^{d-1}.

Condition D. If at least one spherical cap $(\partial Q)_i$ is attached to a component $(\partial Q)_j \subset \partial Q$ then $(\partial Q)_j$ intersects (besides these caps) only flat components of the boundary ∂Q. Moreover, all these flat components are perpendicular to the hyperplane B_i.

Definition 2. The *zone of focusing* of the spherical cap $(\partial Q)_i$ is a polyhedron bounded by the hyperplanes B_i and B_i' and by the hyperplanes which contain flat components of ∂Q intersecting $(\partial Q)_j$, where $(\partial Q)_j \subset \partial Q$ is the regular component to which $(\partial Q)_i$ is attached.

Theorem 1. Let a region $Q \subset \mathbb{R}^d$ satisfy Conditions A–D. Suppose also that the zones of focusing corresponding to different focusing components of the boundary ∂Q do not intersect. Let also each zone of focusing be a regular polyhedron which tiles the whole space $\mathbb{R}^d$. Then the billiard in Q has non-vanishing Lyapunov exponents.

An example of a region satisfying the conditions of Theorem 1 is depicted in Fig. 2.

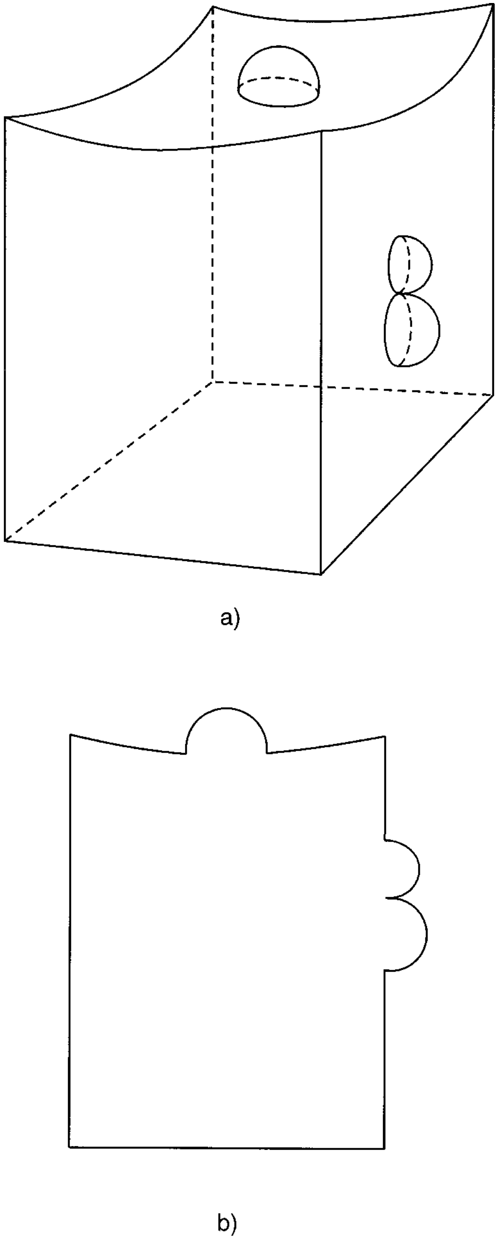

Fig. 2. A billiard region under study (a) and its cross section (b).

Theorem 2. Suppose that all conditions of Theorem 1 are satisfied, but the Condition B is replaced by the Condition B′. Then the billiard in Q is Bernoulli (B-) system.

Corollary. Under the conditions of Theorem 1 a billiard in Q is ergodic, mixing and Kolmogorov (K-) system.

The proofs of Theorems 1 and 2 are completely analogous to the ones in refs. 4 and 5. Only a few trivial additions are needed.

3. ABSOLUTELY FOCUSING MIRRORS

It is quite likely that the conditions of Theorems 1 and 2 are too restrictive, and the ergodic billiards must correspond to much broader classes of d-dimensional ($d \geqslant 3$) regions. In this section we discuss some possible extensions of these theorems and formulate a general conjecture.

The fundamental question[6] in this direction is the following one:

Which focusing components are allowed in the boundary of a chaotic (say, mixing) billiard?

Indeed, dispersing components never belong to a boundary of integrable billiards, while focusing components can belong to the boundary of integrable as well as of chaotic billiards.

This problem is essentially settled for two-dimensional billiards.[10]

Naturally, in the first examples of chaotic billiards with focusing component all these components had constant curvatures, i.e., they were arcs of circles. Then more general classes of focusing components were considered.[11, 12] Finally, the general criterion was introduced.[9] This criterion says that each focusing component of a boundary of a chaotic billiard must be *absolutely focusing*. This notion seems to be new for the geometric optics. It means (in $2D$) that any initially plane wave front of a beam of rays becomes focusing after the series of consecutive reflections from such component. Observe, that a component of a boundary (mirror) Γ is focusing if it focuses all plane fronts just after the first reflection from it. So focusing is a local notion while absolutely focusing is a global one, which deals with the points of the first and of the last (in a series of consecutive) reflections from Γ.

However, the absolute focusing implies "more local" property. It has been shown[9] that in a series of consecutive reflections from an absolutely focusing mirror Γ any flat wave front becomes focusing after any (not just the last one) reflection in a series of consecutive reflections from Γ. (This local property sometimes is more convenient to deal with.[13, 14] Donnay[14] introduced it independently.)

The notion of absolutely focusing surfaces (mirrors) requires the obvious modification in higher ($d \geqslant 3$) dimensions. Consider a planar control surface (wave front) which corresponds to a beam of trajectories which is about to have a series of consecutive reflections from a focusing component (focusing mirror) Γ of a boundary ∂Q of some billiard region Q. Then Γ is called absolutely focusing if after the last reflection in a series any such planar wave front becomes focusing in all directions (in all two-dimensional sections).

Conjecture. A focusing surface (focusing mirror) Γ can be a regular component of the boundary of hyperbolic billiard iff it is absolutely focusing.

It seems that it is still long way to go to the proof or disproof of this conjecture. We will discuss some "more modest" problems in the next section.

It is worthwhile though to justify the notion of absolutely focusing mirrors by presenting the examples of focusing but not absolutely focusing surfaces. Such examples were so far known in $2D$ only.[14]

Consider a spherical cap such that all its central angles are less than $90°$. (Such caps are allowed by the conditions of Theorem 1.) It follows immediately from ref. 4 that such spherical caps are absolutely focusing. On the other hand[4] a simply connected subset of a sphere S^{d-1} ($d \geqslant 3$) with a central angle exceeding $90°$ at least in one two-dimensional section is not absolutely focusing.

4. CONCLUDING REMARKS

We discuss in this section some future problems, which deal with possible generalizations of the presented results and with one new physical model which was inspired by the proof[4, 5] showing how the mechanism of defocusing can work in higher dimensions.

Condition B′ in Theorem 2 is introduced for some technical reasons.[5] It seems that ergodicity (mixing, etc.) should be valid when spherical caps have the internal angles less than $90°$ rather than $60°$. It is possible, however, that there are some subtle differences in the structure of spectra of the two corresponding quantum problems.

The existence of zones of focusing inside a billiard region Q for any spherical cap seems to be essential. Observe though that several spherical caps can be attached to the' same flat component of ∂Q (Fig. 2) and thus their zones of focusing may intersect. The important question is whether or not two focusing components could be attached to the same dispersing component of the boundary.

There are examples of two-dimensional chaotic billiards with only focusing components in the boundary.[1] Also, there are examples of chaotic billiards in convex domains.[3] Numerical experiments[16] seem to demonstrate that such examples may exist in higher dimensions as well. The crucial problem here is to construct examples of chaotic billiards in $d \geqslant 3$ with focusing components which can intersect several regular components of the boundary, rather than to be attached to one component of ∂Q. It is important to emphasize that we consider focusing components rather than semifocusing ones which do not focus along all directions (as e.g., semicylinders). The corresponding convex chaotic billiards can be easily constructed as direct products of two-dimensional chaotic billiards. This idea has already been applied to geodesic flows[17] to produce chaotic flows on high-dimensional product manifolds.

Finally, we will mention the model which was recently introduced to mimic the dynamics of particles in nuclei.[15] Consider a (classical) system of N particles with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} U(|r_i - r_j|) \tag{4}$$

where r_i is a (two-dimensional) position vector of the ith particle and p_i is its momentum. The interaction potential is defined as

$$U(\bar{r}) = \begin{cases} 0 & \text{if} \quad \bar{r} < a \\ \infty & \text{if} \quad \bar{r} \geqslant a \end{cases} \tag{5}$$

Thus, the particles move freely until some pair of particles diverges at the distance a. The total energy, total momentum and total angular momentum are conserved quantities. It is argued[15] that this system is a simple classical model for nuclei or atomic clusters. Numerical experiments[15] show that this system has $2N-4$ positive Lyapunov exponents, i.e., it is completely hyperbolic. Observe, that the potential $U(\bar{r})$ is in a sense dual one to the potential in the system of elastically interacting hard disks. It is well known[1] that the billiard system which corresponds to the hard disks gas is a semi-dispersing billiard. Thus, it is chaotic due to the mechanism of dispersing. The system defined by the Hamiltonian (4) can be reduced to the semi-focusing billiard. Therefore a chaotic dynamics of this system is generated by the mechanism of defocusing.

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